Quantization of fields over de Sitter space by the method of generalized coherent states. II. Spinor field

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Abstract. Connection of the invariant Dirac equation over the de Sitter space with irreducible representations of the de Sitter group is ascertained. The set of solutions of this equation is obtained in the form of the product of two different systems of generalized coherent states for the de Sitter group. Using these solutions the quantized Dirac field over de Sitter space is constructed and its propagator is found. It is a result of action of some de Sitter invariant spinor operator onto the spin zero propagator with an imaginary shift of a mass.

1. Introduction

It seems that in the case of spin 1/2 particles the usual methods are insufficient for consistent construction of a theory of quantized field over the de Sitter (dS) space. Indeed, a lot of papers were concerned with obtaining the solutions of covariant ([1] and references therein) and group theoretical [2] Dirac equation over the dS space by the method of separation of variables, however all these solutions have complicated form which considerably troubles the construction of the theory of quantized field. Only in the little-known paper [3] the summation over one of such a set of solutions was performed; the resulting propagator is not dS-invariant and does not obey the causality principle. In the over hand, in [4] a spinor propagator was found starting from the demands of dS-invariant Dirac equation satisfaction, dS-invariance and the boundary conditions; but the quantized field to which it corresponds was not found; on the contrary, in the Anti-de Sitter space the quantized spinor fields with an invariant propagator was constructed long ago [5].

In the present paper we show that all these troubles may be overcame using the method of generalized coherent states (CS), and build the theory of quantized spinor

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field over the dS space by analogy with the theory of scalar field considered in the part I of this series of paper [9]. The present paper is composed as follows. In section 2 we consider the dS-invariant Dirac equation and show that the corresponding representation of the dS group is irreducible and falls under the classification listed in section 3 of part I. Also we show that this equation admits the reduction to the covariant form by the well simpler way than proposed previously [4]. In section 3 we construct the CS system for the four-spinor representation of the dS group in the form of 4×2-matrices. Solutions of the dS-invariant Dirac equation are the products of these CS and scalar CS obtained in section 4 of part I. In fact, these solutions are the more compact form of the spinor "plane waves" obtained in [6]. The invariance properties of these solutions allow us to construct of them a dS-invariant two-point function and compute it passing to the complexified dS space. In section 4 we construct the quantized spinor field using these solutions; its propagator is expressed by the boundary values of two-point functions obtained in section 3 and coincides with the expression obtained *a priori* in [4], to within the constant multiplier. In section 5 we briefly summarize the results of parts I and II of this series of paper.

2.   The Dirac equation

Introducing the matrices

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \tilde{\gamma}^\mu = -i\gamma^5\gamma^\mu \qquad \tilde{\gamma}^5 = i\gamma^5$$

let us write down the generators of four-spinor representation of the dS group in the five-dimensional form:

$$J^{(s)AB} = \frac{1}{4}[\tilde{\gamma}^A, \tilde{\gamma}^B]. \tag{1}$$

The equalities

$$\tilde{\gamma}^A\tilde{\gamma}^B + \tilde{\gamma}^B\tilde{\gamma}^A = 2\eta^{AB} \tag{2}$$

$$\tilde{\gamma}^A\tilde{\gamma}^B\tilde{\gamma}^C = \eta^{AB}\tilde{\gamma}^C + \eta^{BC}\tilde{\gamma}^A - \eta^{AC}\tilde{\gamma}^B + \frac{1}{2}\varepsilon^{ABCDE}\tilde{\gamma}_D\tilde{\gamma}_E \tag{3}$$

$$\tilde{\gamma}^A\tilde{\gamma}^B\tilde{\gamma}^C\tilde{\gamma}^D = \eta^{AB}\tilde{\gamma}^C\tilde{\gamma}^D + \eta^{BC}\tilde{\gamma}^A\tilde{\gamma}^D - \eta^{AC}\tilde{\gamma}^B\tilde{\gamma}^D +$$
$$+2(\eta^{AD}J^{(s)BC} + \eta^{CD}J^{(s)AB} - \eta^{BD}J^{(s)AC}) - \varepsilon^{ABCDE}\tilde{\gamma}_E \tag{4}$$

hold. With the help of Equations (8) and (9) of Part I and (2)-(4) we obtain

$$R^2C_2^{(s)} = 5/2 \qquad W_A^{(s)} = \frac{3}{4}\tilde{\gamma}_A \qquad R^2C_4^{(s)} = \frac{45}{16}. \tag{5}$$

Comparing the above expression with Equation (10) of Part I we see that it is the representation $\boldsymbol{\pi}_{3/2,3/2}$. We shall choose the standard form of γ-matrices. Then it is easy to show that the explicit form of generators is

$$\boldsymbol{\Pi}^{+(s)} = \frac{1}{R} \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ 0 & 0 \end{pmatrix} \qquad \boldsymbol{\Pi}^{-(s)} = \frac{1}{R} \begin{pmatrix} 0 & 0 \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \tag{6}$$

$$P^{0(s)} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad J_{ik}^{(s)} = -\mathrm{i}\varepsilon_{ikl} \begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix}.$$

We denote matrices of finite transformations as $U(g)$. For their were meeting the same composition properties as the coordinate transformations, it is necessary to take all the parameters of transformations with the sign which is opposite with respect to the scalar representation. Then we obtain

$$U(\Theta_{\pm}(\boldsymbol{a})) = 1 - \boldsymbol{\Pi}^{\pm(s)}\boldsymbol{a}R$$
$$U(\Theta_0(\varepsilon)) = \exp(-P_0^{(s)}\varepsilon).$$

Let us consider the representation $\boldsymbol{\pi}_{3/2,3/2} \otimes \boldsymbol{\nu}_{m,0}$. Its generators are the sum of generators (13) of Part I (*orbital* part) and generators (1) (*spin* part). Then the second-order Casimir operator is equal to

$$C_2 = C_2^{(l)} + C_2^{(s)} - R^{-2} J^{(s)AB} J_{AB}^{(l)}.$$

Denoting $\hat{\nabla}_{\mathrm{dS}} = -R^{-1} J^{(s)AB} J_{AB}^{(l)}$ we obtain

$$C_2 = \square + \frac{\hat{\nabla}_{\mathrm{dS}}}{R} + \frac{5}{2R^2}. \tag{7}$$

To compute the fourth-order Casimir operator we write according to Equation (14) of Part I and (5):

$$RW_A = -\frac{1}{8}\varepsilon_{ABCDE}\tilde{\gamma}^B\tilde{\gamma}^C J^{(l)DE} + \frac{3}{4}\tilde{\gamma}_A.$$

Squaring W_A it is necessary to use formulas (2)-(4). The result obtained

$$C_4 = \frac{3}{4}\square + \frac{3\hat{\nabla}_{\mathrm{dS}}}{4R} + \frac{45}{16R^2}$$

is in agreement with Equation (12) of Part I at $s = 1/2$ and (7). From the second Shur's lemma follows that the operators $\hat{\nabla}_{\mathrm{dS}}$ and $\square$ should have fixed eigenvalues in the irreducible representations. Then using Equation (11) of Part I , (7) and the equality

$$\hat{\nabla}_{\mathrm{dS}}^2 = \frac{1}{4R^2}\tilde{\gamma}^A\tilde{\gamma}^B\tilde{\gamma}^C\tilde{\gamma}^D J_{AB}^{(l)}J_{CD}^{(l)} = \square - 3\hat{\nabla}_{\mathrm{dS}}/R$$

we obtain the quadratic equation for eigenvalues of $\hat{\nabla}_{\mathrm{dS}}$. Solving it yields

$$\hat{\nabla}_{\mathrm{dS}} = -2R^{-1} \pm \mathrm{i}\mu \tag{8}$$
$$\square = -\mu^2 \mp \mathrm{i}R^{-1}\mu - 2R^{-2}$$

where $\mu^2 = m^2 - \frac{1}{4R^2}$. As $m^2 > 1/4R^2$ (see section 3 of part I), then μ is a real number. The appearance of two signs indicates that two identical irreducible representations had appeared:

$$\boldsymbol{\nu}_{m,0} \otimes \boldsymbol{\pi}_{3/2,3/2} = 2\boldsymbol{\nu}_{m,1/2}.$$

Using Equation (13) of Part I we can write

$$\hat{\nabla}_{\mathrm{dS}} = \Gamma^\mu \partial_\mu \qquad \Gamma^\mu = \chi\gamma^\mu + \frac{1}{2R}[\gamma^\mu, \gamma_\nu]x^\nu.$$

Choosing the representation which corresponds to the lower sign in (8) we finally obtain the group theoretical Dirac equation over the dS space:

$$i\Gamma^\mu \partial_\mu \psi - (\mu - \frac{2i}{R})\psi = 0. \tag{9}$$

This was well known previously out of the context of dS group irreducible representations [7].

The above equation admits the transformation into the covariant form. To this end let us perform the transformation $\Psi = V\psi$, where

$$V = (1 - \varepsilon_\mu \varepsilon^\mu)^{-1/2}(1 + \gamma_\mu \varepsilon^\mu) \qquad \varepsilon^\mu = \frac{x^\mu}{R(\chi + 1)}.$$

Then (9) passes into

$$iV\Gamma^\mu V^{-1}(\partial_\mu \Psi + (V\partial_\mu V^{-1})\Psi) - (\mu - 2iR^{-1})\Psi = 0. \tag{10}$$

It is easy to show that

$$V\Gamma^\mu V^{-1} = e^\mu_{(\nu)}\gamma^\nu$$
$$\partial_\mu + V\partial_\mu V^{-1} = \mathcal{D}_\mu - \frac{1}{2R}\gamma_\mu \tag{11}$$

where $e^{(\mu)}_\nu$ is the vierbein which is orthonormal with respect to the metric (3) of Part I:

$$e^{(\mu)\nu} = \eta^{\mu\nu} + \frac{x^\mu x^\nu}{R^2(\chi + 1)}$$

and $\mathcal{D}_\mu$ is the spinor covariant derivative

$$\mathcal{D}_{(\mu)} = e^\nu_{(\mu)}\partial_\nu - \frac{1}{2}J^{(s)\nu\rho}G_{\nu\rho\mu} \qquad G_{\nu\rho\mu} = e^\sigma_{(\nu);\kappa}e_{(\rho)\sigma}e^\kappa_{(\mu)} = \frac{1}{R^2(\chi + 1)}(x_\nu\eta_{\mu\rho} - x_\rho\eta_{\nu\mu}).$$

Then putting together (10)-(11) we finally obtain

$$i\gamma^\mu e^\nu_{(\mu)}\mathcal{D}_\nu \Psi = \mu\Psi.$$

Another more complicated way of transformation of dS-invariant Dirac equation to the covariant one was proposed in [4].

3. Spinor coherent states

Let us denote the constant 4×2-matrices as A, A', A'' and define over such matrices the weak equivalence relation $\sim$ and the strong one $\simeq$ as

$$A' \sim A'' \Leftrightarrow A' = A''B \qquad B \in GL(2, \mathbb{C})$$
$$A' \simeq A'' \Leftrightarrow A' = A''B \qquad B \in SU(2).$$

Also, let us define the product of two 4×2-matrices A' and A'' as $A'\overline{A''}$, where the upper line denotes the Dirac conjugation. Consider the left action of four-spinor representation of the dS group over these matrices: $g: A \mapsto U(g)A$. It is easy to show that the matrices

$$|+\rangle = \begin{pmatrix} I_2 \\ 0_2 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0_2 \\ I_2 \end{pmatrix}$$

(where I_2 is the unit 2×2-matrix) are invariant under transformations which belong to the subgroups $\mathcal{K}^{\pm} \equiv \mathcal{T}^{\pm}\textcircled{S}(\mathcal{T}^0 \otimes \mathcal{R})$ to within the weak equivalence relation. In the terms of the strong equivalence relation we have

$$U(h)|\pm\rangle \simeq (\alpha_{\boldsymbol{v}}^{\pm}(h))^{-1/2}|_{\boldsymbol{v}=\boldsymbol{o}}|\pm\rangle \qquad h \in \mathcal{K}^{\pm}.$$

From the other hand, it is easily seen that the subgroups $\mathcal{K}^{\pm}$ are stability subgroups of the vector $\boldsymbol{w} = \boldsymbol{o}$ concerning the conformal action (19) of Part I of the dS group. This allows us to identify the $SO(4,1)/\mathcal{K}^{\pm}$ space with the space $\mathbb{R}^3$ of vectors $\boldsymbol{w}$. As the lifting from the $SO(4,1)/\mathcal{K}^{\pm}$ space to the dS group we shall take the transformation which transforms the origin into the point $\boldsymbol{w}$:

$$SO(4,1)/\mathcal{K}^{\pm} \ni \boldsymbol{w} \mapsto g_{\boldsymbol{w}} = \Theta_{\mp}(-\boldsymbol{w}) \in SO(4,1).$$

Then the CS system for the $SO(4,1)/\mathcal{K}^{\pm}$ space being dS-invariant to within the weak equivalence relation is

$$|\boldsymbol{w}\pm\rangle = U(g_{\boldsymbol{w}})|\pm\rangle$$
$$|\boldsymbol{w}+\rangle = \begin{pmatrix} I_2 \\ \sigma\boldsymbol{w} \end{pmatrix} \qquad |\boldsymbol{w}-\rangle = \begin{pmatrix} -\sigma\boldsymbol{w} \\ I_2 \end{pmatrix}.$$

With the help of Equation (1) of Part I the transformation properties of these vectors with respect to the strong equivalence relation may be written as

$$U(g_1)|\boldsymbol{w}\pm\rangle \simeq (\alpha_{\boldsymbol{v}}^{\pm}(g_{\boldsymbol{w'}}^{-1}g_1g_{\boldsymbol{w}}))^{-1/2}|_{\boldsymbol{v}=\boldsymbol{o}}|\boldsymbol{w}_{g_1}\pm\rangle \qquad g \in \mathcal{G}. \qquad (12)$$

As the transformations $T_{\sigma}^{\pm}(g)$ compose a representation of the dS group then

$$\alpha_{\boldsymbol{v}}^{\pm}(g_2g_1) = \alpha_{\boldsymbol{v}}^{\pm}(g_2)\alpha_{\boldsymbol{v'}}^{\pm}(g_1) \qquad g_1, g_2 \in \mathcal{G} \qquad \boldsymbol{v'} = \boldsymbol{v}_{g_2^{-1}}.$$

Then using the above expression and Equation (1) of Part I we get

$$\alpha_{\boldsymbol{v}}^{\pm}(g_1) = \alpha_{\boldsymbol{v'}}^{\pm}(g_{\boldsymbol{w'}}^{-1}g_1g_{\boldsymbol{w}}) \qquad \boldsymbol{v'} = \boldsymbol{v}_{g_{\boldsymbol{w'}}^{-1}} \qquad \boldsymbol{w'} = \boldsymbol{w}_{g_1}.$$

Putting $\boldsymbol{v} = \boldsymbol{w}'$ in the above expression, we can rewrite the transformation properties (12) as

$$(\alpha_{\boldsymbol{w}}^{\pm}(g))^{1/2} U(g) |\boldsymbol{w}_{g^{-1}}\pm\rangle \simeq |\boldsymbol{w}\pm\rangle \qquad g \in \mathcal{G}. \tag{13}$$

It is easy to show that the equalities

$$(\gamma \cdot k_{\boldsymbol{w}} \mp 1) |\boldsymbol{w}\pm\rangle = 0 \tag{14}$$

$$|\boldsymbol{w}\pm\rangle\langle \boldsymbol{w} \pm | = \frac{1 - \mathrm{w}^2}{2}(\gamma \cdot k_{\boldsymbol{w}} \pm 1) \tag{15}$$

are correct. Now let us construct the 4×2-matrix functions

$$\Phi_{\boldsymbol{w}}^{(1/2)\pm}(x) = \Phi_{\boldsymbol{w}}^{(0)\pm}(x; \sigma_0 - 1/2)|\boldsymbol{w}\pm\rangle.$$

Using (14) we obtain that they obey (9):

$$(\mathrm{i}\hat{\nabla} - \mu + 2\mathrm{i}R^{-1})\Phi_{\boldsymbol{w}}^{(1/2)\pm}(x) = 0.$$

These solutions are much simpler that these obtained by the method of separation of variables [1, 2].

From the transformation properties (21) of Part I and (13) it follows that under transformations from the dS group the functions $\Phi_{\boldsymbol{w}}^{(1/2)\pm}(x)$ transform just as the functions $\Phi_{\boldsymbol{w}}^{(0)\pm}(x; \sigma_0)$, to within the constant matrix transformation:

$$\Phi_{\boldsymbol{w}}^{(1/2)\pm}(x_g) \simeq (\alpha_{\boldsymbol{w}}^{\pm}(g))^{\sigma_0} U(g) \Phi_{\boldsymbol{w}'}^{(1/2)\pm}(x) \tag{16}$$

where $\boldsymbol{w}' = \boldsymbol{w}_{g^{-1}}$. As the inversion $\boldsymbol{w} \mapsto -\boldsymbol{w}/w^2$ yields

$$\Phi_{-\boldsymbol{w}/w^2}^{(1/2)\pm}(x) \simeq -i(-w^2)^{-\sigma_0}\Phi_{\boldsymbol{w}}^{(1/2)\mp}(x)$$

then the functions $\Phi_{\boldsymbol{w}}^{(1/2)\pm}(x)$ and $\Phi_{\boldsymbol{w}}^{(1/2)\pm}(x)$ yield the same two-point function. Let us define its as follows:

$$\frac{1}{8}\mathcal{W}^{(1/2)}(\overset{1}{x}, \overset{2}{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{w}\, \Phi_{\boldsymbol{w}}^{(1/2)+}(\overset{1}{x})\overline{\Phi}_{\boldsymbol{w}}^{(1/2)+}(\overset{2}{x}) \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{w}\, \Phi_{\boldsymbol{w}}^{(1/2)-}(\overset{1}{x})\overline{\Phi}_{\boldsymbol{w}}^{(1/2)-}(\overset{2}{x}).$$

From (16) follows that the above functions are dS-invariant in the sense that at $g \in \mathcal{G}$

$$\mathcal{W}^{(1/2)}(\overset{1}{x}_g, \overset{2}{x}_g) = U(g)\mathcal{W}^{(1/2)}(\overset{1}{x}, \overset{2}{x})\overline{U}(g).$$

Using (15) it is easy to show that

$$\mathcal{W}^{(1/2)}(\overset{1}{x}, \overset{2}{x}) = \frac{1}{2}\int_{S^3} \frac{\mathrm{d}^3\boldsymbol{l}}{l^5} \left(\frac{\overset{1}{x}^0 + l^a \overset{1}{x}^a}{R}\right)^{-\mathrm{i}\mu R - 2} \left(\frac{\overset{2}{x}^0 + l^a \overset{2}{x}^a}{R}\right)^{\mathrm{i}\mu R - 2} (\gamma^0 + \gamma\boldsymbol{l} + l^5).$$

As the functions $\varphi_k^{(1/2)\pm}(\zeta; \lambda)$ inherit the analyticity properties of functions $\varphi_k^{(0)\pm}(\zeta; \sigma_0)$ over the complexified dS space, then, as in the spin zero case, the functions $\mathcal{W}^{(1/2)}(\overset{1}{x}, \overset{2}{x})$

converge at $(\overset{1}{\zeta}, \overset{2}{\zeta}) \in \mathcal{D}^+ \times \mathcal{D}^-$. Choosing the points according to Equation (23) of Part I and using the equality

$$\int_0^\pi d\theta \, \sin^2\theta \cos\theta (\cosh v + \sinh v \cos\theta)^{-i\mu R - 2}$$

$$= \frac{\pi}{\sinh v} \left({}_2F_1\left(1 - \frac{i\mu R}{2}, \frac{1}{2} + \frac{i\mu R}{2}; 2; -\sinh^2 v\right) \right.$$

$$\left. - \cosh v \, {}_2F_1\left(1 + \frac{i\mu R}{2}, \frac{1}{2} - \frac{i\mu R}{2}; 2; -\sinh^2 v\right) \right)$$

we obtain

$$\mathcal{W}^{(1/2)}(\overset{1}{\zeta}, \overset{2}{\zeta}) = \frac{\pi^2 e^{-\pi\mu R}}{\mu - iR^{-1}} \tilde{\gamma}_A \overset{1}{\zeta}{}^A \left(i\hat{\nabla}_{dS} - \mu + \frac{i}{R}\right) {}_2F_1\left(2 - i\mu R, 1 + i\mu R; 2; \frac{1 - \rho}{2}\right) \gamma^5 \tag{17}$$

where the operator $\hat{\nabla}_{dS}$ acts onto the coordinates $\overset{1}{\zeta}$.

4. Spinor field over de Sitter space

To construct a quantized spinor field, let us use the equality

$$R^{-1}\tilde{\gamma}^A x_A \left(i\hat{\nabla}_{dS} - \mu + 2iR^{-1}\right) R^{-1}\tilde{\gamma}^B x_B = i\hat{\nabla}_{dS} + \mu + 2iR^{-1}.$$

From here follows that if the function ψ obey the Dirac equation (9), then the function $\tilde{\gamma}^A x_A \psi$ obey the same equation with the opposite sign of μ. Then the functions

$$\tilde{\Phi}_{\boldsymbol{w}}^{(1/2)\pm}(x) = R^{-1}\tilde{\gamma}^A x_A \Phi_{\boldsymbol{w}}^{(0)\pm}(x; \sigma_0^* - 1/2)|\boldsymbol{w}\pm\rangle$$

obey Equation (9). Let us introduce also two sets of fermionic creation-annihilation operators $b^{(\pm)}(\boldsymbol{w})$ and $b^{(\pm)\dagger}(\boldsymbol{w})$, which at the same time are the matrices of dimensionality 2×1 and 1×2, respectively, and obey the anticommutation relations

$$\{b^{(\pm)}(\boldsymbol{w}), b^{(\pm)\dagger}(\boldsymbol{w}')\} = \delta(\boldsymbol{w}, \boldsymbol{w}') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{18}$$

and all other anticommutators vanish. Then we can construct the quantized spinor field as

$$\phi^{(1/2)}(x) = \int_{\mathbb{R}^3} d^3\boldsymbol{w} \left(\Phi_{\boldsymbol{w}}^{(1/2)+}(x)b^{(+)}(\boldsymbol{w}) + \tilde{\Phi}_{\boldsymbol{w}}^{(1/2)-}(x)b^{(-)}(\boldsymbol{w})\right). \tag{19}$$

Using (17) it is easy to show that the two-point function which corresponds to the solutions $\tilde{\Phi}_{\boldsymbol{w}}^{(1/2)\pm}(x)$ is equal to

$$\int_{\mathbb{R}^3} d^3\boldsymbol{w} \, \tilde{\Phi}_{\boldsymbol{w}}^{(1/2)\pm}(\overset{1}{\zeta}) \overline{\tilde{\Phi}}_{\boldsymbol{w}}^{(1/2)\pm}(\overset{2}{\zeta}) =$$

$$R^{-2}\tilde{\gamma}^A \overset{1}{\zeta}_A \mathcal{W}^{(1/2)}(\overset{2}{\zeta}, \overset{1}{\zeta})(\gamma^5 \tilde{\gamma}^B \overset{2}{\zeta}_B \gamma^5) = -\mathcal{W}^{(1/2)}(\overset{1}{\zeta}, \overset{2}{\zeta}). \tag{20}$$

Further, the hypergeometric functions in the r.h.s. of (17) differ from $\mathcal{W}^{(0)}(\overset{1}{\zeta},\overset{2}{\zeta})$ only by the constant multiplier and the imaginary shift of mass. Then computing the difference of its values on the edges of the cut $z \in [1,+\infty)$ we can use the results of section 5 of part I. Passing to the boundary values Equation (20) yields

$$R^{-2}\tilde{\gamma}^A \overset{1}{x}_A \mathcal{W}^{(1/2)+}(\overset{2}{x},\overset{1}{x})(\gamma^5 \tilde{\gamma}^B \overset{2}{x}_B \gamma^5) = -\mathcal{W}^{(1/2)-}(\overset{1}{x},\overset{2}{x})$$

which is analogous to Equation (27) of Part I for the spin zero case. Then using Equation (28) of Part I for the spin 1/2 propagator

$$\{\phi^{(1/2)}_\alpha(\overset{1}{x}), \overline{\phi}^{(1/2)}_\beta(\overset{2}{x})\} \equiv G^{(1/2)}_{\alpha\beta}(\overset{1}{x},\overset{2}{x}) = \mathcal{W}^{(1/2)+}_{\alpha\beta}(\overset{1}{x},\overset{2}{x}) - \mathcal{W}^{(1/2)-}_{\alpha\beta}(\overset{1}{x},\overset{2}{x})$$

where $\alpha,\beta = 1,\dots,4$ are spinor indices, we finally obtain

$$G^{(1/2)}(\overset{1}{x},\overset{2}{x}) = \frac{i\pi^2}{\mu - iR^{-1}}(1 + e^{-2\pi\mu R})\varepsilon(\overset{1}{x}{}^0 - \overset{2}{x}{}^0)\tilde{\gamma}_A \overset{1}{x}{}^A$$

$$\times \left(i\hat{\nabla}_{\mathrm{dS}} - \mu + \frac{i}{R}\right)\theta\left(-\frac{1+G}{2}\right){}_2F_1\left(2 - i\mu R, 1 + i\mu R; 2; \frac{1+G}{2}\right)\gamma^5$$

where the operator $\hat{\nabla}_{\mathrm{dS}}$ acts onto the coordinates $\overset{1}{x}$. The above expression coincides with the solution of Cauchy problem for the Dirac equation over the dS space obtained in [4]. The only difference is that our method do not allow us to find the behavior of propagator over the "light cone" $G = -1$.

5. Concluding remarks

To summarize the results of the present series of papers, we can say that the CS method allow us to quantize massive spin 0 and 1/2 fields over the dS space in the uniform way. Both in the spin zero case and in the spin 1/2 one the starting-point is the invariant wave equations which correspond to irreducible representations of the dS group. The solutions of these equations are constructed from CS for the dS group; in the spin zero case the dS-invariant Klein-Gordon equation is satisfied by the scalar CS itself. In the spin 1/2 case the solutions of dS-invariant Dirac equation are constructed from two different CS systems which correspond to different representations of the dS group and different stationary subgroups. Both in the spin zero case and in the spin 1/2 one these sets of solutions possess the same transformation properties under the dS group, with difference that the constant matrix transformation is added in the spin 1/2 case.

From these sets of solutions we can construct the two-point functions $\mathcal{W}^{(s)}(\overset{1}{x},\overset{2}{x})$ which have the following properties:

(i) dS-invariance:

$$\mathcal{W}^{(1/2)}(\overset{1}{x}_g,\overset{2}{x}_g) = U_s(g)\mathcal{W}^{(1/2)}(\overset{1}{x},\overset{2}{x})\overline{U}_s(g)$$

where $U_s(g)$ is the identical representation at $s = 0$ and the four-spinor representation at $s = 1/2$.

(ii) Causality:

$$\mathcal{W}^{(s)}(\overset{1}{x}, \overset{2}{x}) = \mathcal{W}^{(s)}(\overset{2}{x}, \overset{1}{x}) \qquad \overset{1}{x}_A \overset{2}{x}^A > -R^2.$$

(iii) Regularized function $\mathcal{W}^{(s)}(\overset{1}{x}, \overset{2}{x})$ is the boundary value of the function $\mathcal{W}^{(s)}(\overset{1}{\zeta}, \overset{2}{\zeta})$ which is analytic in certain domain of the complexified dS space.

For the spin zero case the above properties were proved in [8]; but also in this case the CS method gives the sufficient simplification since the property 1 is found almost obvious. In the theorem 4.1 of the mentioned paper the property of *positive definiteness*

$$\int\int \frac{\mathrm{d}^4\,\overset{1}{x}}{\overset{1}{x}^5} \frac{\mathrm{d}^4\,\overset{2}{x}}{\overset{2}{x}^5} \mathcal{W}^{(0)}(\overset{1}{x}, \overset{2}{x}) f(\overset{1}{x}) f^*(\overset{2}{x}) > 0$$

was proved for any functions $f(x)$ such that this integral is meaningful, where $\mathcal{W}^{(0)}(\overset{1}{x}, \overset{2}{x})$ is considered in the sense of boundary values. For the spin $1/2$ case this property may be proved in the completely analogous way since

$$\int\int \frac{\mathrm{d}^4\,\overset{1}{x}}{\overset{1}{x}^5} \frac{\mathrm{d}^4\,\overset{2}{x}}{\overset{2}{x}^5} \overline{\psi}(\overset{1}{x}) \mathcal{W}^{(1/2)}(\overset{1}{x}, \overset{2}{x}) \psi(\overset{2}{x}) = \int_{\mathbb{R}^3} \mathrm{d}^3 w \left| \int \frac{\mathrm{d}^4 x}{x^5} \overline{\psi}(x) \Phi_{\boldsymbol{w}}^{(1/2)-}(x) \right|^2.$$

Defining the creation-annihilation operators so that they possess the necessary commutation relations, we can construct quantized fields $\phi^{(s)}(x)$; the propagators of these fields are equal to

$$[\phi^{(s)}(\overset{1}{x}), \overline{\phi}^{(s)}(\overset{2}{x})]_{\pm} = \mathcal{W}^{(s)}(\overset{1}{x}, \overset{2}{x}) - \mathcal{W}^{(s)}(\overset{2}{x}, \overset{1}{x})$$

and therefore are dS-invariant and causal automatically.

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